

SOME REMARKS ON GLAISHER-RAMANUJAN TYPE INTEGRALS

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ABSTRACT. Some integrals of the Glaisher-Ramanujan type are established in a more general form than in previous studies. As an application we prove some Ramanujan-type series identities, as well as a new formula for the Dirichlet beta function at the value $s = 3$.

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1. INTRODUCTION

In a paper of Glasser [5], the work of Glaisher [4] and Ramanujan [7] was extended to present further evaluations of the integral

$$(1.1) \quad \int_0^\infty \eta^n(ix) f(x) dx,$$

for integers $n \geq 1$, and particular elementary functions $f(x)$. Here, as usual, the Dedekind-eta function is given by

$$(1.2) \quad \eta(ix) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q = e^{-2\pi x}$, for real $x > 0$. For some commentary on integrals of this type we refer the reader to [5, 7] and references therein.

In this note, we restrict our attention to the case $f(x) = e^{-bx} \cos(cx)$, and $f(x) = e^{-bx} \sin(cx)$, in (1.1). This provides a refinement to integrals like (8) of [5] and (19)–(28) in [4]. That is, we shall prove

Theorem 1.1. *For $b > 0$ and $c > 0$ we have*

$$(1.3) \quad \int_0^\infty \eta^3(i4x/\pi) e^{-b^2x} \sin(cx) dx = \frac{\pi}{4c} \frac{\sinh(\frac{\pi}{2}A(b, c)) \sin(\frac{\pi}{2}B(b, c))}{\sinh^2(\frac{\pi}{2}A(b, c)) + \cos^2(\frac{\pi}{2}B(b, c))},$$

$$(1.4) \quad \int_0^\infty \eta^3(i4x/\pi) e^{-b^2x} \cos(cx) dx = \frac{\pi}{4} \frac{\cosh(\frac{\pi}{2}A(b, c)) \cos(\frac{\pi}{2}B(b, c))}{\cosh^2(\frac{\pi}{2}A(b, c)) - \sin^2(\frac{\pi}{2}B(b, c))},$$

where $2A(b, c)^2 = \sqrt{b^4 + c^2} + b^2$, and $2B(b, c)^2 = \sqrt{b^4 + c^2} - b^2$.

Note that $c \rightarrow 0$ of (1.4) gives (14) of [5].

From here we can easily extend the work of Glasser to obtain evaluations of integrals involving $\eta^n(iax)\eta^k(ibx)$, for integers $n, k \geq 1$, and $a, b \in \mathbb{R}$. Throughout this paper we define

$$(1.5) \quad \chi(n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod{2}, \\ 1, & \text{if } n \equiv 1 \pmod{4}, \\ -1, & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Theorem 1.2. *For $c > 0$ we have*

$$(1.6) \quad \int_0^\infty \eta^6(i4x/\pi) \sin(cx) dx = \frac{\pi}{2c} \sum_{n \geq 1} \chi(n)n \frac{\sinh(\frac{\pi}{2}A(n, c)) \sin(\frac{\pi}{2}B(n, c))}{\sinh^2(\frac{\pi}{2}A(n, c)) + \cos^2(\frac{\pi}{2}B(n, c))},$$

$$(1.7) \quad \int_0^\infty \eta^6(i4x/\pi) \cos(cx) dx = \frac{\pi}{4} \sum_{n \geq 1} \chi(n)n \frac{\cosh(\frac{\pi}{2}A(n, c)) \cos(\frac{\pi}{2}B(n, c))}{\cosh^2(\frac{\pi}{2}A(n, c)) - \sin^2(\frac{\pi}{2}B(n, c))}.$$

Unfortunately, as we observe, the right sides of (1.4) and (1.5) are not expressible in terms of elementary functions like those of Glasser's [5] and Glaisher's [4]. We can also obtain other examples using the same procedure as Glasser, by appealing to different theta series. In particular, by Euler's identity [1, p.575] we have the following.

Theorem 1.3. *For $c > 0$ we have*

$$(1.8) \quad \int_0^\infty \eta^3(i4x/\pi) \eta(i12x/\pi) \sin(cx) dx = \frac{\pi}{4} \sum_{n \in \mathbb{Z}} (-1)^n \frac{\cosh(\frac{\pi}{2}A(6n+1, c)) \cos(\frac{\pi}{2}B(6n+1, c))}{\cosh^2(\frac{\pi}{2}A(6n+1, c)) - \sin^2(\frac{\pi}{2}B(6n+1, c))}.$$

2. THE PROOF

To prove Theorem 1.1 we require some simple series evaluations that we were unable to find in the literature. Our methods are similar to those of [4] and we only require some known integral evaluations and the Poisson summation formula for Fourier sine transforms [2, p.257]. If $f(x)$ is a continuous, real-valued function with bounded total variation on $[a, b]$ then

$$(2.1) \quad \sum_{a \leq n \leq b} \chi(n)f(n) = \sum_{n \geq 1} \chi(n) \int_a^b f(x) \sin(\pi xn/2) dx.$$

By I. S. Gradshteyn and I. M. Ryzhik [6, p.428], we have

$$(2.2) \quad \int_0^\infty \frac{x \sin(ax) dx}{(x^2 + b^2)^2 + c^2} = \frac{\pi}{2c} e^{-aA(b,c)} \sin(aB(b,c)),$$

$$(2.3) \quad \int_0^\infty \frac{x(x^2 + b^2) \sin(ax) dx}{(x^2 + b^2)^2 + c^2} = \frac{\pi}{2} e^{-aA(b,c)} \cos(aB(b,c)),$$

where $A(b, c)$ and $B(b, c)$ are as in Theorem 1.1, and $a > 0$, $b > 0$, and $c > 0$.

Lemma 2.1. *For $b > 0$ and $c > 0$ we have*

$$(2.4) \quad \sum_{n \geq 1} \chi(n) \frac{n}{(n^2 + b^2)^2 + c^2} = \frac{\pi}{4c} \frac{\sinh(\frac{\pi}{2}A(b, c)) \sin(\frac{\pi}{2}B(b, c))}{\sinh^2(\frac{\pi}{2}A(b, c)) + \cos^2(\frac{\pi}{2}B(b, c))},$$

$$(2.5) \quad \sum_{n \geq 1} \chi(n) \frac{n(n^2 + b^2)}{(n^2 + b^2)^2 + c^2} = \frac{\pi}{4} \frac{\cosh(\frac{\pi}{2}A(b, c)) \cos(\frac{\pi}{2}B(b, c))}{\cosh^2(\frac{\pi}{2}A(b, c)) - \sin^2(\frac{\pi}{2}B(b, c))}.$$

Proof. For (2.4) apply (2.1) with $f(x) = \frac{x}{(x^2 + b^2)^2 + c^2}$ and invoke (2.2). For (2.5) apply (2.1) with $f(x) = \frac{x(x^2 + b^2)}{(x^2 + b^2)^2 + c^2}$ and invoke (2.3).

For (1.3), we use identity (12) of [5] (with x replaced by $x4/\pi$) to find

$$\begin{aligned} & \int_0^\infty \eta^3(ix4/\pi) e^{-b^2x} \sin(cx) dx \\ &= \sum_{n \geq 1} \chi(n) n \int_0^\infty e^{-(b^2 + n^2)x} \sin(cx) dx \\ &= c \sum_{n \geq 1} \chi(n) \frac{n}{(n^2 + b^2)^2 + c^2}. \end{aligned}$$

□

By (2.4) of Lemma 2.1 the proof is straightforward. It is not difficult to prove (1.4).

The only difference is that we appeal to the Fourier cosine transform and employ (2.5).

3. AN APPLICATION TO RAMANUJAN-TYPE SERIES

In Ramanujan's notebook [2] we find the amazing formula for $\zeta(\frac{1}{2})$: If $x > 0$, then

$$\begin{aligned} & \sum_{n \geq 1} \frac{1}{e^{n^2x} - 1} = \frac{\pi^2}{6x} + \frac{1}{2} \sqrt{\frac{\pi}{x}} \zeta\left(\frac{1}{2}\right) + \frac{1}{4} \\ (3.1) \quad & + \sqrt{\frac{\pi}{2x}} \sum_{n \geq 1} \frac{1}{\sqrt{n}} \left(\frac{\cos(\frac{\pi}{4} + 2\pi\sqrt{\pi n/x}) - e^{-2\pi\sqrt{\pi n/x}} \cos(\frac{\pi}{4})}{\cosh(2\pi\sqrt{\pi n/x}) - \cos(2\pi\sqrt{\pi n/x})} \right). \end{aligned}$$

Several authors have produced generalizations of this formula [2, 3, 8, 9]. Authors [9] obtain a formula for the Dirichlet L -function for $\bar{\chi}$, the primitive Dirichlet character modulo q , at $s = 1$. In this section we will obtain a formula for the special value $s = 3$ of the Dirichlet beta function [1]

$$(3.2) \quad \beta(s) = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^s}.$$

Theorem 3.1. *For $z > 0$ we have*

$$(3.3) \quad \begin{aligned} & \frac{\pi}{8} + \sum_{n \geq 1} \frac{\chi(n)}{n(e^{n^2 z} - 1)} \\ &= \frac{\beta(3)}{z} + \frac{1}{2\pi} \sum_{n \geq 1} \frac{\sinh(\frac{\pi}{2} \sqrt{\frac{n\pi}{z}}) \sin(\frac{\pi}{2} \sqrt{\frac{n\pi}{z}})}{n(\cosh(\pi \sqrt{\frac{n\pi}{z}}) + \cos(\pi \sqrt{\frac{n\pi}{z}}))}. \end{aligned}$$

Proof. Under the same hypothesis as for sine transforms for the Poisson summation formula, we have [2, p.252]

$$(3.4) \quad \sum'_{a \leq n \leq b} f(n) = \int_a^b f(x) dx + 2 \sum_{n \geq 1} \int_a^b f(x) \cos(\pi 2xn) dx,$$

with the additional condition that the prime on the sum indicates only $\frac{1}{2}f(a)$ is counted if a is finite, and similarly for b . We choose the function $(x, z > 0)$

$$f(x) = \sum_{n \geq 1} \chi(n) \frac{e^{-n^2 x z}}{n},$$

which has the range $\mathbb{R}^+ = (0, \infty)$. $f(x)$ has bounded variation since, over any closed interval $I \subset \mathbb{R}^+$, there exists a constant M such that $\sum_{i \geq 1}^n |f(x_i) - f(x_{i-1})| < M$, for all partitions of I .

Glaisher [4, eq.(23)] offers

$$(3.5) \quad \sum_{n \geq 1} \chi(n) \frac{e^{-n^2 z}}{n} = \frac{1}{2} \int_0^\infty \cos(zx) \frac{\sinh(\frac{\pi}{2} \sqrt{\frac{x}{2}}) \sin(\frac{\pi}{2} \sqrt{\frac{x}{2}}) dx}{x(\cosh(\pi \sqrt{\frac{x}{2}}) + \cos(\pi \sqrt{\frac{x}{2}}))},$$

or

$$(3.6) \quad \int_0^\infty \cos(zx) \sum_{n \geq 1} \chi(n) \frac{e^{-n^2 z \alpha}}{n} dz = \frac{1}{2} \frac{\sinh(\frac{\pi}{2} \sqrt{\frac{x}{2\alpha}}) \sin(\frac{\pi}{2} \sqrt{\frac{x}{2\alpha}})}{x(\cosh(\pi \sqrt{\frac{x}{2\alpha}}) + \cos(\pi \sqrt{\frac{x}{2\alpha}}))},$$

for $\alpha > 0$. Choosing $a = 0$ and $b = \infty$ in (3.4) with our choice of $f(x)$, we get the theorem after noting

$$\int_0^\infty f(x) dx = \frac{\beta(3)}{z},$$

and $f(0) = \frac{\pi}{4}$. □

Here we were able to observe some further integral evaluations and an interesting application. It remains a challenge to the reader to find specific instances when integrals of the type (1.1) involving $\eta^n(ax)\eta^k(bx)$ can be evaluated for natural numbers $n, k \geq 1$.

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